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Generalised hyperKähler manifolds in string theory

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Abstract: We discuss the notion of generalised hyperKähler structure in the context of string theory and discuss examples of this geometry.

Keywords: Superstring Vacua, Sigma Models.

Contents

1. Introduction

Strings, being extended, require a generalisation of the geometry of spacetime that goes beyond the conventional notion. This is at the heart of novel and beautiful ideas such as mirror symmetry. More recently there has been a lot of advances in understanding the motion of a string in the 'compact' internal space with various fluxes that are possible in string theory. This has prompted the investigation of an appropriate geometrical framework. One such idea is that of the generalised complex geometry introduced by Hitchin [1, 2] and developed further by Gualtieri [3], in which both the tangent as well as the cotangent spaces of the manifold are considered together. As a matter of fact, this occurs naturally in string theory when there is a magnetic flux of the Neveu-Schwarz (NS) B-field. Demanding supersymmetry under this condition requries that the left- and right-moving modes of the string perceive different complex (or Kähler or hyperKähler) structures [4]–[9]. In purely geometrical terms, this turns out to be equivalent to a generalised complex (or Kähler or hyperKähler) structure [3] (see also [10]). Essentially these two descriptions are the lagrangian and hamiltonian approaches to the string sigma model. A number of papers [11]–[26] have analysed various aspects of generalised geometry in the context of string theory in recent times (for a recent review, see [27]).

In this short note, we will elaborate on the generalised hyperKähler structure alluded to by Hitchin [1, 2] and discussed by others [3, 28] and mention an explicit example of a generalised hyperkahler manifold found in string theory. In view of the comments above, it is clear that these ought to be alternative descriptions of string backgrounds that can be equivalently described in terms of left- and right-movers. Our example is indeed a reformulation of the familiar Neveu-Schwarz fivebrane [29] in terms of generalised geometry.

It should also be mentioned that string theory allows for fluxes of various other fields. In particular, there are generalised gauge form fields of all degrees from Ramond sector.

Although some work has been done, finding a geometrical framework for diverse flux vacua remains a challenging open problem.

In the following, we briefly recapitulate some essential aspects of generalised complex and Kähler structures. This is then extended to generalised hyperKähler structure. We provide an example of generalised hyperKähler geometry from string theory. In the end, we propose possible constructions that further generalise the idea.

Note added: the paper [30], which appeared while we have been studying this problem, has some overlap with our work. However, the focus of [30] is on the analysis of the worldsheet sigma model and supersymmetry, while we have provided explicit examples which satisfy these conditions.

2. Generalised complex and Kähler geometry

A generalised complex structure extends the notion of the usual complex structure to the sum of the tangent and cotangent bundles $T\mathcal{M} \oplus T^*\mathcal{M}$ of a manifold $\mathcal{M}$.

An almost generalised complex structure in an open neighbourhood of $p \in \mathcal{M}$ is a linear map

$$\Im : T\mathcal{M} \oplus T^*\mathcal{M} \to T\mathcal{M} \oplus T^*\mathcal{M}, \text{ such that } \Im^2 = -\mathbf{1}. \tag{2.1}$$

This extends to a generalised complex structure when $\Im$ can be defined consistently over $\mathcal{M}$ leading to integrability conditions similar to the ordinary one [1, 3]. The dimension of the manifold $\mathcal{M}$ must be even. Trivial examples are ordinary complex structure I, together with its transpose on $T^*\mathcal{M}$:

$$\Im_I = \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix}, \tag{2.2}$$

and symplectic structure, along with its inverse:

$$\Im_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{2.3}$$

In fact, a generalised complex structure is locally a product of complex and symplectic structures [3].

A generalised Kähler structure requires two commuting generalised complex structures $\Im_1$ and $\Im_2$, such that

$$\Im_1 \, \Im_2 = -\,\mathcal{G} = -\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}, \tag{2.4}$$

where $\mathcal{G}$ is a positive definite generalised metric[1] on $T\mathcal{M} \oplus T^*\mathcal{M}$. The motivation is clearly to extend the idea of a hermitian Kähler manifold.

This structure first appeared in the physics literature in the work of Gates *et al* [4], where they analysed the requirement for additional global supersymmetry for the sigma

[1] In a more general situation, the generalised metric is a B-transform of the above [3], in which case one has the B-transforms of the $\Im$'s.

model describing a string propagating in the presence of the Neveu-Schwarz B-field in a target space K. This is a necessary requirement for spacetime supersymmetry. They showed that the general form of the additional supersymmetry variation of the (1,1) superfields Φ^i for the coordinates, is of the form:

$$\delta\Phi^i = \epsilon^+ f^i_{+j} D_+ \Phi^j + \epsilon^- f^i_{-j} D_- \Phi^j, \tag{2.5}$$

where $D_\pm$ are the worldsheet super-covariant derivatives. The tensors $f_\pm$ are a pair of almost complex structures on the target manifold K for the left- and right-movers respectively; further, they have to satisfy the constraint

$$\nabla_i^\pm f^j_{\pm k} = \partial_i f^j_{\pm k} + \Gamma^j_{\pm im} f^m_{\pm k} - \Gamma^m_{\pm ik} f^j_{\pm m} = 0. \tag{2.6}$$

In the above $\Gamma^i_{\pm jk}$ are the connections seen by the left- and right movers

$$\Gamma^i_{\pm jk} = \Gamma^i_{jk} \pm g^{im} H_{mjk}, \tag{2.7}$$

where Γ^i_{jk} are Levi-Civita connections derived from the metric and $H = dB$. This leads to what was called a bi-hermitian geometry for K, namely distinct Kähler structures for the left- and right-movers. Gualtieri [3] has shown that the bi-hermitian geometry of [4] can equivalently be written as a generalised Kähler geometry.

3. Generalised hyperKähler geometry: strings in a hyperKähler manifold with H-flux

A generalised hyperKähler structure naturally carries the idea of a generalised Kähler structure one step further and requires six generalised complex structures $\Im^\pm_a$, $(a = 1, 2, 3)$. Recall that an ordinary hyperKähler structure has three complex structures I_a $(a = 1, 2, 3)$ transforming as an SU(2) triplet and three compatible symplectic structures $\omega_a = gI_a$. The $\Im^\pm_a$'s satisfy the following algebra that can be motivated from the hyperKähler case:

$$\begin{aligned} \Im^+_a \Im^+_b &= -\delta_{ab} + \epsilon_{abc} \Im^+_c, \\ \Im^-_a \Im^-_b &= -\delta_{ab} + \epsilon_{abc} \Im^+_c, \\ \Im^+_a \Im^-_b &= -\delta_{ab}\mathcal{G} + \epsilon_{abc} \Im^-_c, \\ \Im^-_a \Im^+_b &= -\delta_{ab}\mathcal{G} + \epsilon_{abc} \Im^-_c, \end{aligned} \tag{3.1}$$

where, ϵ_{abc} is a totally antisymmetric symbol and $\mathcal{G}$ is as in eq. (2.4).

As we have seen above, for a string moving in a manifold with the metric g in the presence of a H-flux, there is a torsion term that modifies the connection for the left- and right-movers. In case of (4,4) supersymmetry on the worldsheet, there are three complex structures $f^\pm_a$ $(a = 1, 2, 3)$ in each sector. The generalised hyperKähler structure is obtained straightforwardly by following from the prescription of ref. [3]:

$$\begin{aligned} \Im^+_a &= \begin{pmatrix} \frac{1}{2}(f^+_a + f^-_a) & \frac{1}{2}(f^+_a - f^-_a)\, g^{-1} \\ \frac{1}{2} g\,(f^+_a - f^-_a) & -\frac{1}{2}\left(f^{t+}_a + f^{t-}_a\right) \end{pmatrix}, \\ \Im^-_a &= \begin{pmatrix} \frac{1}{2}(f^+_a - f^-_a) & \frac{1}{2}(f^+_a + f^-_a)\, g^{-1} \\ \frac{1}{2} g\,(f^+_a + f^-_a) & -\frac{1}{2}\left(f^{t+}_a - f^{t-}_a\right) \end{pmatrix}, \end{aligned} \tag{3.2}$$

where $f_a^{t\pm}$ denote the transpose of $f_a^{\pm}$. It is easy to see that these satisfy the algebra (3.1).

4. Generalised hyperKähler geometry of the NS5-brane

We will show that the Neveu-Schwarz 5-brane solution found in ref. [29] provides an example of generalised hyperKähler geometry. The NS5-brane was given as a soliton solution in the supergravity approximation, however, at the same time a worldsheet conformal field theory description established it as an exact solution of string theory. Suppose the NS5-brane extends along the coordinates $(x^1 \cdots x^5)$. The space labelled by $(x^6 \cdots x^9)$ is the transverse space K. In the supergravity approximation, the metric g of K is such that in the near horizon limit the geometry is that of a cylinder with an S^3 base. There is an H-flux through the S^3 and also a linear dilaton along the length of the cylinder. Explicitly, the background is given by:

$$\begin{aligned} g_{ij} &= e^{2\phi}\delta_{ij}, \quad i,j\ldots = 6,\cdots,9, \\ H_{ijk} &= -\epsilon_{ijk}{}^{m}\partial_m\phi, \qquad (4.1) \\ \nabla^2 e^{2\phi} &= 0, \qquad (4.2) \end{aligned}$$

where ϕ is the dilaton field. It was also shown that there is an exact (4,4) superconformal field theory on the worldsheet, for details see [29].

As mentioned earlier, the torsion term modifies the connection. Therefore, the left- and right-movers perceive different hyperKähler structures as follows [29]:

$$\begin{aligned} f_1^+ &= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad f_1^- = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \\ f_2^+ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad f_2^- = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad (4.3) \\ f_3^+ &= \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad f_3^- = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \end{aligned}$$

The above, which are an extension of [4] to the hyperKähler case, may be called a bi-hyperKähler structure. The generalised hyperKähler structure is then given by the eq. (3.2). Clearly, these satisfy the algebra (3.1). Thus we see that the (transverse space) of the NS5-brane of ref. [29] is a natural example of generalised hyperKähler manifold.

It should be possible to find other examples of generalised hyperKähler geometries, which are solutions to the string equations of motion.

5. Further generalisation

Finally, we would like to propose a construction for a *compact* 'generalised hyperKähler manifold'. In going from one 'coordinate chart' to another, we will now allow for S-duality transformations in addition to T-duality used in generalised geometry.

Consider type IIB string theory. There are two different 2-form fields B_{NS} and B_R originating in the NS-NS and RR sectors respectively. There is also an SL(2,$\mathbf{Z}$) duality symmetry under which these transform as a doublet. If we combine any of the SL(2,$\mathbf{Z}$) transform of the B-fields with the metric, it will be seen as a generalised complex structure by a suitable SL(2,$\mathbf{Z}$) transform of the fundamental string. Let us start with an elliptically fibred K3 manifold. Now turn on an appropriate SL(2,$\mathbf{Z}$) transform of the B-fields following the monodromy of the torus fibre, so that these are consistent globally over the entire manifold. The singularities around which there are monodromies, correspond to different (p,q)-5-branes, which are patched together so that the transverse space is a *compact* 'generalised hyperKähler manifold'. This construction, if it can be made globally consistent, can also be extended to other dimensions to obtain, for example, generalised Calabi-Yau manifolds.

The above proposal is, therefore, in the spirit of F-theory, U-manifold [32, 33] and 'generalised geometry' as proposed by Hull [34, 35].

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